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BOUNDARY VALUE PROBLEM FOR A PARABOLIC-HYPERBOLIC EQUATION LOADED BY THE FRACTIONAL ORDER INTEGRAL OPERATOR

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Abstract. In the paper a boundary value problem is studied for a loaded parabolic-hyperbolic equation with three lines of type change, when the loaded part contains an integro-differentiation operator of fractional order. Using the properties of fractional order operators in the sense of Riemann-Liouville and the Green's function, the unique solvability of the stated problem is proved.

Keywords: loaded equation, local problem, Fredholm integral equation with shift, Green's function.

AMS Subject Classification: 35M10, 35M12, 35K15.

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1 Introduction

The main results related to the application of integro-differentiation operators in the theory of differential equations belong to scientists (Jrbashyan et al., 1968). Jrbashyan and Nersesyan (1968) investigated the unambiguous solvability of the Cauchy problem for a differential equation with one variable, with the operator of generalized Riemann-Liouville integro-differentiation, in currently called by their names. In this case, the considering problem under consideration is equivalently reduced to solve an integral equation of the Volterra type, and the solution is expressed using special functions of the Mittag-Leffler type. In Jrbashyan, (1966), the properties of the Mittag-Leffler function are investigated and the first boundary value problem is solved in the case when the order of the equation is less than two.

In Gorenflo (2000); Kilbas (2004); Pskhu (2003), the Cauchy problem and boundary value problems with the fractional order operator of Riemann-Liouville and Caputo, which have great values in the construction of mathematical models in diffusion processes, were investigated for the diffusion-wave equation. To study these problems, the method of reduction to a system of equations of a smaller order, the Laplace transform, and also the method of the Green function are applied.

Operator methods for solving boundary value problems for fractional order equations were considered in the works of Ashurov at al. (2016); Luchko at al. (1999); Shinaliev at al. 2012).

The Laplace transform and the Fourier transform were also used in the works of Kilbas at al. (2006); Gekkieva (2000) to construct fundamental solutions of fractional order diffusion and

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wave equations with Caputo and Riemann-Liouville derivatives, as well as to solve boundary value problems and the Cauchy problem.

In Nakhushev (1959) proposed a number of problems of a new type that have entered the mathematical literature named boundary value problems with displacement, as it turned out, which, as it turned out, are closely related to loaded differential equations (Nakhushev, 2006).

It was found out that many very important problems of mathematical physics and biology (Nakhushev, 1983; Nakhushev, 1995), especially the problems of long-term forecasting and regulation of groundwater, the problems of heat and mass transfer with finite velocity, the movement of a slightly compressible fluid surrounded by a porous medium, lead to boundary value problems for loaded partial differential equations.

It should be noted that fractal objects based on non-local mathematical models of physical and biological processes contain loaded equations with fractional derivatives. In turn, mixed boundary value problems for partial differential equations are reduced to loaded differential equations with integro-differential operators of integer or fractional order. In the process of studying mixed-type equations with a load, What it the exist principles of extremum and existence theorems, as well as the methods of classical theory, cannot be applied directly. It is important to prove the uniqueness of the solution of the considering problems for loaded equations. In this regard, in the works of Islamov at al. (1996; Islamov at al. (2011), boundary value problems for loaded equations were considered, resulting in poorly studied Volterra and Fredholm integral equations with a shift.

Boundary value problems for loaded equations of the second order of hyperbolic, parabolic, hyperbolic-parabolic and elliptic-parabolic types, with one line of type change are considered in Genaliev (2001); Kozhanov (2004); Sabitov (2015); Juraev (1979); Mikhlin (1959). Some boundary value problems for matrix factorizations of the Helmholtz equation are considered in Juraev (2019); Juraev (2020); Juraev at al. (2022).

In Islomov at al. (2015a;b;c); Islamov at al. (2016); Islamov at al. (2021) local and non-local problems for loaded equations of mixed type of the second order with three lines of type change were studied.

In work Abdullaev (2020) using the method of lines, a numerical solution of a boundary value problem with respect to a loaded parabolic equation with nonlocal boundary conditions was investigated, and calculation formulas were obtained and an algorithm for solving the problem was given. In Ubaydullayev (2020) the inverse problem for a mixed loaded equation with a Riemann-Liouville operator in a rectangular domain was studied. A uniqueness criterion is established. A solution to the problem is constructed as the sum of a series in terms of eigenfunctions of the corresponding one-dimensional spectral problem. It is proved that the unique solvability of the inverse problem essentially depends on the choice of the boundary of the rectangular domain. An example is constructed in which the inverse problem with homogeneous conditions has a nontrivial solution. In Yuldashev at al. (2020) an analogue of the Gellerstedt problem for a loaded third-order parabolic-hyperbolic equation in an infinite three-dimensional domain is studied. The main method for studying this Gellerstedt problem is the Fourier transform. Based on the Fourier transform, the problem under consideration is reduced to a flat analogue of the spectral Gellerstedt problem with a spectral parameter. The uniqueness of the solution of this problem is proved by a new extremal principle for loaded equations of mixed type of the third order.

This paper is devoted to the formulation and study of a local boundary value problem for a loaded parabolic-hyperbolic equation with three lines of changes type when the loaded part contains an integro-differential operator of fractional order.

2 Problem statement A_μ

Consider the equation

$$0=\begin{cases} u_{xx}-u_y-\mu_0 D_{0x}^{\alpha_0}u(x,0) & (x,y)\in\Omega_0,\\ u_{xx}-\ u_{yy}-\mu_1 D_{0\xi}^{\alpha_1}u(\xi,0), & (x,y)\in\Omega_1,\\ u_{xx}-\ u_{yy}-\mu_2 D_{0\eta}^{\alpha_2}u(0,\eta), & (x,y)\in\Omega_2,\\ u_{xx}-\ u_{yy}-\mu_3 D_{0\zeta}^{\alpha_3}u(1,\zeta), & (x,y)\in\Omega_3\end{cases} \tag{1}$$

in the domain

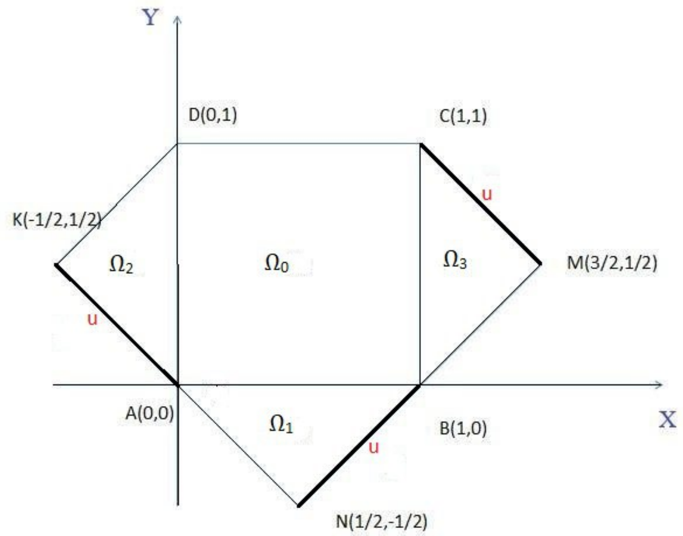

Figure 1: The area describing the equation (1)

$\Omega=\sum\limits_{j=0}^{3}\Omega_j\cup AB\cup BC\cup DA$, where $\xi=x+y\,,\quad \eta=y-x$, and Ω_0 *is* a region bounded by segments $AB,\ \ BC,\ \ CD,\ \ DA$ of straight lines $y=0,\ \ x=1,y=1,\ \ x=0$, respectively;

Ω_1 –a characteristic triangle bounded by the segment AB of the axis Ox and two characteristics $AN:x+y=0$, $BN:x-y=1$ of the equation (1), coming from the points $A(0,0)$ and $B(1,0)$, intersecting at the point $N\,(0,5\,;\ -0,5)$;

Ω_2 –characteristic triangle bounded by the segment AD of the axis Oy and two characteristics $AK:x+y=0,DK:\ \ y-x=1$ of the equation (1), going out from the points $A(0,0)$ and $D(0,1)$, intersecting at the point $K\,(-0,5\,;\ 0,5)$;

Ω_3 –characteristic triangle bounded by the segment BC and two characteristics CM : $x+y=2$, $BM:\ \ y-x=1$ of the equation (1), going out from the points $B(1,0)$ and $C(1,1)$, intersecting at the point $M\,(1,5\,;\ 0,5)$. In the equation (1) $\mu_j\ (j=\overline{1,3})$ are given real numbers, and

$$\mu_j\geq 0,\qquad 0\leq\alpha_j<1,\ \ (j=\overline{0,3})\,. \tag{2}$$

Here $D_{k\,x^l}^{c}f\,(x)$ is the fractional order integro-differentiation operator $|c|$ in the Riemann-Liouville sense (Kilbas at al., 2006):

$$D_{kx^l}^{c}f(x)=\begin{cases}\frac{sgn(x-k)}{\Gamma(-c)}\int\limits_k^x f(t)\left|x^l-t^l\right|^{-c-1}dt^l, & c<0,\\ f(x), & c=0,\\ \left[sgn(x-k)\right]^{1+h}\left(\frac{d^{h+1}}{dx^{l(h+1)}}\right)^{h+1}D_{kx^l}^{c-(h+1)}f(x), & c>0,\end{cases} \tag{3}$$

where $f(x),\ f^{(h)}(x) \in L_1(a,b),\ a<b<\infty,\ l = const > 0,\ h$ is the integer part of $c\ \ (c>0)$, and $D^c_{kx^l} \equiv D^c_{kx^l}$ by $x>k$, $D^c_{kx^l} \equiv D^c_{x^l k}$ by $x<k$.

Remark 1: Note that a nonlocal boundary value problem for a model equation of mixed type can be reduced to a local problem for loaded equations of the form (1)(Nakhushev, 1995). Let's introduce the notation:

$$J_1 = \{(x,y): 0<x<1,\ y=0\},$$

$$J_2 = \{(x,y): 0<y<1,\ x=0\}, J_3 = \{(x,y): 0<y<1,\ x=1\},$$

$$\Omega_1^* = \Omega_1 \cup J_1 \cup \Omega_0, \quad \Omega_2^* = \Omega_2 \cup J_2 \cup \Omega_0 \cup J_3 \cup \Omega_3,$$

$$W = \left\{u:\ u \in C\left(\bar{\Omega}\right),\ u_y \in C\left(\Omega_2^*\right) \cap C\left(\Omega_0 \cup J_1\right) \cap C\left(\Omega_1 \cup J_1\right),\right.$$

$$\left. u_x \in C\left(\Omega_1^*\right) \cap C\left(\Omega_0 \cup J_2 \cup J_3\right) \cap C\left(\Omega_2 \cup J_2\right) \cap C\left(\Omega_3 \cup J_3\right)\right\}.$$

Definition 1. *We call the solution of equation* (1) *as the regular solution that a function* $u(x,y)$ *have continuous partial derivatives up to the second order, inclusively, and inverting its to the identity in the domains* $\Omega_j\ (j=\overline{1,3})$.

Problem A_μ. Find a regular solution of the equation (1) in the domain Ω from the class W satisfying the boundary conditions

$$u(x,y)|_{NB} = \varphi_1(x), \qquad \frac{1}{2} \le x \le 1, \tag{4}$$

$$u(x,y)|_{AK} = \varphi_2(y), \qquad 0 \le y \le \frac{1}{2}, \tag{5}$$

$$u(x,y)|_{MC} = \varphi_3(y), \quad \frac{1}{2} \le y \le 1, \tag{6}$$

and on the lines of changing the type of bonding conditions

$$u_y(x,+0) = a_1(x) u_y(x,-0) + b_1(x), \quad (x,0) \in J_1, \tag{7}$$

$$u_x(-0,y) = a_2(y) u_x(+0,y) + b_2(y), \quad (0,y) \in J_2, \tag{8}$$

$$u_x(1-0,y) = a_3(y) u_x(1+0,y) + b_3(y), \quad (1,y) \in J_3, \tag{9}$$

where $\varphi_1(x), \varphi_2(y), \varphi_3(y),\ a_1(x),\ a_2(y),\ a_3(y),\ b_1(x),\ b_2(y),\ b_3(y)$ are given functions, and

$$a_1(x),\ b_1(x) \in C\left(\bar{J}_1\right) \cap C^1\left(J_1\right), \tag{10}$$

$$a_j(y),\ b_j(y) \in C\left(\bar{J}_j\right) \cap C^2\left(J_j\right), \quad a_j(y) \ne 0,\ \forall y \in \bar{J}_j,\ (j=2,3), \tag{11}$$

$$\varphi_1(1) = 0, \varphi_1(x) \in C^1\left[\frac{1}{2},1\right] \cap C^2\left(\frac{1}{2},1\right), \tag{12}$$

$$\varphi_2(0) = 0, \varphi_3(y) \in C^1\left[\frac{1}{2},1\right] \cap C^2\left(\frac{1}{2},1\right), \varphi_2(y) \in C^1\left[0,\frac{1}{2}\right] \cap C^2\left(0,\frac{1}{2}\right). \tag{13}$$

Remark 1. *The method of research and the novelty of this work is, firstly, to find a representation of the solution of the problem* A_μ *for a loaded equation, and secondly, using this representation of the problem* A_μ *we can equivalently reduce it to Volterra integral equations of the second kind with a weak singularity. The kernels and the right side of the obtained integral equations are also studied.*

3 Deriving the main functional relations

The solution of the Cauchy problem with conditions

$$u\left(x,-0\right)=\tau_1\left(x\right),\quad (x,0)\in\bar{J}_1,\qquad u_y\left(x,-0\right)=\nu_1^-\left(x\right),\quad (x,0)\in J_1, \tag{14}$$

for the equation (1) in the area of Ω_1 express in the form

$$u\left(x,y\right)=\frac{1}{2}\left[\tau_1\left(x+y\right)+\tau_1\left(x-y\right)\right]+\frac{1}{2}\int\limits_{x-y}^{x+y}\nu_1^-\left(t\right)dt+$$

$$+\frac{\mu_1}{4}\int\limits_{x-y}^{x+y}D_{0\xi}^{\alpha_1}\tau_1(\xi)d\xi\int\limits_{\xi}^{x-y}d\eta. \tag{15}$$

Using (4) from (15) we get

$$u|_{y=x-1}=\varphi_1\left(x\right)=\frac{1}{2}\tau_1\left(2x-1\right)+\frac{1}{2}\tau_1\left(1\right)+\frac{1}{2}\int\limits_{1}^{2x-1}\nu_1^-\left(t\right)dt+$$

$$+\frac{\mu_1}{4}\int\limits_{1}^{2x-1}D_{0\xi}^{\alpha_1}\tau_1(\xi)d\xi\int\limits_{\xi}^{1}d\eta,$$

or assuming $2x-1=z$ and changing z to x in the last relation, and then differentiating by the variable x taking into account (3), we get a functional relation between $\tau_1(x)$ and $\nu_1^-(x)$, brought from the area of Ω_1 on J_1 :

$$\nu_1^-\left(x\right)+\tau_1^{'}\left(x\right)+\frac{\mu_1(1-x)}{2\Gamma(1-\alpha_1)}\int\limits_0^x(x-t)^{-\alpha_1}\tau_1'\left(t\right)dt=\varphi_1'\left(\frac{x+1}{2}\right). \tag{16}$$

Similarly, using the solution of the Cauchy problem with initial data

$$u\left(-0,y\right)=\tau_2\left(y\right),\quad (0,y)\in\bar{J}_2,\quad u_x\left(-0,y\right)=\nu_2^-\left(y\right),\quad (0,y)\in J_2, \tag{17}$$

$$(u\left(1-0,y\right)=\tau_3\left(y\right),\ (1,y)\in\bar{J}_3,u_x\left(1-0,y\right)=\nu_3^-\left(y\right),(1,y)\in J_3), \tag{18}$$

for the equation (1) in the domain $\Omega_2(\Omega_3)$taking into account (3), (5) and (6), we obtain a functional relationship between $\tau_2(x)\left(\tau_3(x)\right)$ and $\nu_2^-(x)\left(\nu_3^-(x)\right)$, brought from the area of $\Omega_2\left(\Omega_3\right)$ on $J_2\left(J_3\right)$:

$$\nu_2^-\left(y\right)-\tau_2^{'}\left(y\right)+\frac{\mu_2 y}{2\Gamma(1-\alpha_2)}\int\limits_0^y(y-t)^{-\alpha_2}\tau_2'\left(t\right)dt\;=-\varphi_2'\left(\frac{y}{2}\right), \tag{19}$$

$$\left(\nu_3^-\left(y\right)-\tau_3'\left(y\right)+\frac{\mu_3(1-y)}{2\Gamma(1-\alpha_3)}\int\limits_0^y(t-y)^{-\alpha_3}\tau_3'\left(t\right)dt=-\varphi_3'\left(\frac{y+1}{2}\right)\right). \tag{20}$$

Passing to limit at $y\to+0$ in the equation (1), taking into account the function class $u(x,y)$ of the problem A_μ and $\varphi_1(0)=\tau_1\left(0\right)=0$, we get the functional relationship between $\tau_1'(x)$ and $\nu_1^+(x)$, brought from the area Ω_0 to J_1:

$$\tau_1'\left(x\right)-\int\limits_0^x\nu_1^+\left(t\right)dt-\frac{\mu_0}{\Gamma(2-\alpha_0)}\int\limits_0^x(x-t)^{1-\alpha_0}\tau_1'\left(t\right)dt=\tau_1'(0), \tag{21}$$

where $\tau_1'(0)$ is an unknown constant that must be defined. Solving the first boundary value problem with the conditions $u\left(x,+0\right)=\tau_1\left(x\right),(x,0)\in\bar{J}_1,\ u\left(+0,y\right)=\tau_2\left(y\right),(0,y)\in\bar{J}_2$, $u\left(1+0,y\right)=\tau_3\left(y\right),(1,y)\in\bar{J}_3$ for the equation (1) in the area of Ω_0 has the form (Juraev, 1979):

$$u(x,y)=u_0(x,y)+\int\limits_0^y d\eta\int\limits_0^1 P_1\left(x,y;\ \xi,\ \eta\right)\ u_0(\xi,\eta)d\xi, \tag{22}$$

where

$$u_0\left(x,y\right)=\int\limits_0^y G_\xi\left(x,y;\ 0,\eta\right)\tau_2\left(\eta\right)d\eta+\int\limits_0^y G_\xi\left(x,y;\ 1,\eta\right)\tau_3\left(\eta\right)d\eta+$$

$$+\int\limits_0^1 G\left(x,y;\ \xi,\ 0\right)\tau_1\left(\xi\right)d\xi, \tag{23}$$

$$G(x,y;\,\xi,\eta)=\frac{1}{2\sqrt{\pi\left(y-\eta\right)}}\left\{exp\left\{-\frac{\left(x-\xi\right)^2}{4\left(y-\eta\right)}\right\}-\ exp\left\{-\frac{\left(x+\xi\right)^2}{4\left(y-\eta\right)}\right\}\right\}+$$

$$+\frac{1}{2\sqrt{\pi\left(y-\eta\right)}}\sum\limits_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left\{exp\left\{-\frac{\left(x-\xi+2n\right)^2}{4\left(y-\eta\right)}\right\}-exp\left\{-\frac{\left(x+\xi+2n\right)^2}{4\left(y-\eta\right)}\right\}\right\}, \tag{24}$$

$G(x,y;\,\xi,\eta)$ is the Green function of the first boundary value problem for the equation $u_{xx}-u_y=0$, and $P_1(x,y;\xi,\eta)$ kernel resolvent $\mu_0 D_{0\xi}^{-\alpha_0}\tau(\xi)\cdot G\left(x,y;\ \xi,\ \eta\right)$.

Differentiating (22) by x, and then using the properties of the Green function taking into account $\varphi_2(0)=\tau_2\left(0\right)=0,\ \varphi_1(1)=\tau_3\left(0\right)=0,\ \lim\limits_{z\to 0}z^{-\sigma}e^{-\frac{1}{z}}=0,\quad(\sigma>0)$, we obtain, respectively, a functional relation between $\tau_2(y)$ $(\tau_3(y))$ and $\nu_2^+(y)$ $\left(\nu_3^+(y)\right)$, brought from the area of Ω_0 to $J_2\,(J_3)$:

where

$$\nu_2^+\left(y\right)=-\frac{1}{\sqrt{\pi}}\int\limits_0^y\frac{\tau_2'\left(\eta\right)}{\sqrt{y-\eta}}d\eta\ -\frac{1}{\sqrt{\pi}}\int\limits_0^y\frac{P_2\left(y,\,\eta\right)}{\sqrt{y-\eta}}\tau_2'\left(\eta\right)d\eta+F_2\left(y,\tau_3',\tau_1\right), \tag{25}$$

$$\left(\nu_3^+\left(y\right)=\frac{1}{\sqrt{\pi}}\int\limits_0^y\frac{\tau_3'\left(\eta\right)}{\sqrt{y-\eta}}d\eta+\frac{1}{\sqrt{\pi}}\int\limits_0^y\frac{P_3\left(y,\eta\right)}{\sqrt{y-\eta}}\tau_3'\left(\eta\right)d\eta+F_3\left(y,\tau_2',\tau_1\right)\right), \tag{26}$$

where

$$u_x\left(+0,y\right)=\nu_2^+\left(y\right),\quad(0,y)\in J_2, u_x\left(1+\,0,y\right)=\nu_3^+\left(y\right),\quad(1,y)\in J_3,$$

$$P_2\left(y,\,\eta\right)=2\sum\limits_{n=1}^{+\infty}e^{-\frac{n^2}{y-\eta}},P_3\left(y,\,\eta\right)=e^{-\frac{1}{y-\eta}}+\sum\limits_{n=1}^{+\infty}e^{-\frac{(n+1)^2}{y-\eta}}+\frac{1}{2}P_2\left(y,\,\eta\right), \tag{27}$$

$$F_2\left(y,\tau_3',\tau_1\right)=\frac{2}{\sqrt{\pi}}\int\limits_0^y\frac{\tau_3'\left(\eta\right)}{\sqrt{y-\eta}}e^{-\frac{1}{4(y-\eta)}}\,d\eta+\frac{2}{\sqrt{\pi}}\int\limits_0^y\frac{\tau_3'\left(\eta\right)}{\sqrt{y-\eta}}\sum\limits_{n=1}^{+\infty}e^{-\frac{(1+2n)^2}{4(y-\eta)}}\,d\eta+$$

$$+\frac{1}{2\sqrt{\pi y}}\int\limits_0^1\sum\limits_{n=-\infty}^{+\infty}e^{-\frac{4n^2+\xi^2}{4y}}\left[\frac{\xi}{y}ch4\xi n-\frac{2n}{y}sh4\xi n)\right]\ \tau_1\left(\xi\right)d\xi+$$

$$+\frac{1}{2\sqrt{\pi}\Gamma(1-\alpha_0)}\int\limits_0^y \frac{d\eta}{\sqrt{y-\eta}}\times$$

$$\times\int\limits_0^1 \sum_{n=-\infty}^{+\infty} e^{-\frac{4n^2+\xi^2}{4(y-\eta)}}\left[\frac{\xi}{y-\eta}ch4\xi n-\frac{2n}{y-\eta}sh4\xi n\right]d\xi\int\limits_0^\xi(\xi-t)^{-\alpha_0}\tau_1'(t)\,dt, \tag{28}$$

$$F_3\left(y,\tau_2',\tau_1\right)=-\frac{2}{\sqrt{\pi}}\int\limits_0^y\frac{\tau_2'(\eta)}{\sqrt{y-\eta}}e^{-\frac{1}{4(y-\eta)}}d\eta-\frac{2}{\sqrt{\pi}}\int\limits_0^y\frac{\tau_2'(\eta)}{\sqrt{y-\eta}}\sum_{n=1}^{+\infty}e^{-\frac{(1+2n)^2}{4(y-\eta)}}d\eta+$$

$$+\frac{1}{2\sqrt{\pi y}}\int\limits_0^1\sum_{n=-\infty}^{+\infty}e^{-\frac{(1+2n)^2+\xi^2}{4y}}\left[\frac{\xi}{y}ch2\xi(1+2n)-\frac{1+2n}{y}sh2\xi(1+2n)\right]\tau_1(\xi)\,d\xi+$$

$$+\frac{1}{2\sqrt{\pi}\Gamma(1-\alpha_0)}\int\limits_0^y\frac{d\eta}{\sqrt{y-\eta}}\int\limits_0^1\sum_{n=-\infty}^{+\infty}e^{-\frac{(1+2n)^2+\xi^2}{4(y-\eta)}}\times$$

$$\times\left[\frac{\xi}{y-\eta}ch2\xi(1+2n)-\frac{1+2n}{y-\eta}sh2\xi(1+2n)\right]d\xi\times$$

$$\times\int\limits_0^\xi(\xi-t)^{-\alpha_0}\tau_1'(t)\,dt. \tag{29}$$

4 Studying of the problem A_μ

Theorem 1. *If the conditions* (2), (9)-(12) *are fulfilled and*

$$a_1(x)<0,\ \ \forall x\in\bar{J}_1, \tag{30}$$

then there is a single regular solution of the problem A_μ are hold true in the domain of Ω.

Proof. Excluding $\nu_1^-(x)$ from the relations (16) and (21) taking into account (7) after some calculations, we obtain an integral equation with respect to $\tau_1'(x)$:

$$\tau_1'(x)+\int\limits_0^x Q_1(x,t)\tau_1'(t)\,dt=\tau_1'(0)+\Phi_1(x),\quad (x,0)\in J_1, \tag{31}$$

where

$$Q_1(x,t)=a_1(t)+\frac{\mu_1(1-x)}{2\Gamma(1-\alpha_1)}\int\limits_t^x\frac{(1-z)a_1(z)}{(z-t)^{\alpha_1}}dz-\frac{\mu_0(x-t)^{1-\alpha_0}}{\Gamma(2-\alpha_0)}, \tag{32}$$

$$\Phi_1(x)=\int\limits_0^x a_1(t)\varphi_1'\left(\frac{t+1}{2}\right)dt-\int\limits_0^x b_1(t)dt. \tag{33}$$

Using of (2), (10), (12) from (32) and (33) taking into account the class W it follows that

$$|Q_1(x,t)|\le c_1\ \ 0\le x\le 1,\quad 0\le t\le 1; \tag{34}$$

and

$$\Phi_1(x)\in C\left(\bar{J}_1\right)\bigcap C^2\left(J_1\right). \tag{35}$$

Thus, using of (34) and (35), the equation (31) is a Volterra integral equation of the second kind. According to the theory of Volterra integral equations (Mikhlin, 1959) we conclude that the integral equation (31) is uniquely solvable in the class

$$C\left(\bar{J}_1\right)\bigcap C^2\left(J_1\right)$$

and its solution is given by the formula

$$\tau_1'\left(x\right)=\tau_1'(0)+\Phi_1(x)-\int\limits_0^x Q_1^*(x,t)\left[\tau_1'(0)+\Phi_1(t)\right]dt,\quad (x,0)\in J_1, \tag{36}$$

where $Q_1^*(x,t)$ is the resolvent of the kernel $Q_1(x,t)$. Integrating (36) from 0 to x taking into account $\tau_1(0)=\varphi_2(0)=0$, we have

$$\tau_1\left(x\right)=\int\limits_0^x\left[\tau_1'(0)+\Phi_1(t)\right]dt-\int\limits_0^\xi dt\int\limits_0^t Q_1^*(t,z)\left[\tau_1'(0)+\Phi_1(z)\right]dz,\ \ (x,0)\in\bar{J}_1. \tag{37}$$

Now after putting in (37) $x=1$ taking into account $\varphi_1(1)=\tau_1\left(1\right)=0$ we find the unknown constant $\tau_1'(0)$:

$$\tau_1'\left(0\right)=-\frac{\int\limits_0^1\left[\Phi_1\left(t\right)+\int\limits_0^t Q_1^*(t,z)\Phi_1\left(z\right)dz\right]dt}{1-\int\limits_0^1 dt\int\limits_0^t Q_1^*(t,z)dz}. \tag{38}$$

Using (30), it follows from (32) that $Q_1(x,t)<0,\ \ \forall\, x,t\in[0,1]$. Therefore, the resolvent of the kernel $Q_1(x,t)$ is also negative, i.e.

$$Q_1^*(x,t)<0,\quad \forall\, x,t\in[0,1]\,.$$

So the denominator of the formula (38) for any $0\le x\le 1,\quad 0\le t\le 1$ does not vanish, i.e.

$$1-\int\limits_0^1 dt\int\limits_0^t Q_1^*(t,z)dz>0.$$

Using (34) and (35) of (37), taking into account (38), we conclude that

$$\tau_1(x)\in C^1\left(\bar{J}_1\right)\bigcap C^2\left(J_1\right). \tag{39}$$

By supplying (36) to (16) taking into account (10), (12), (39), we define the function $\nu_1^-\left(x\right)$ from class $\nu_1^-\left(x\right)\in C\left(\bar{J}_1\right)\cap C^1\left(J_1\right)$.

After excluding $\nu_j^-\left(y\right)$ and $\nu_j^+\left(y\right)$ from the relations (18), (19) and (22), (23) taking into account (8), (9) we obtain systems of integral equations with respect to $\tau_j'\left(y\right),\ (j=2,3)$:

$$\tau_2'\left(y\right)-\int\limits_0^y Q_2(y,t)\tau_2'\left(t\right)dt=\int\limits_0^y M_2(y,t)\tau_3'\left(t\right)dt+\Phi_2(y,\tau_1), \tag{40}$$

$$\tau_3'\left(y\right)-\int\limits_0^y Q_3(y,t)\tau_3'\left(t\right)dt=\int\limits_0^y M_3(y,t)\tau_2'\left(t\right)dt+\Phi_3(y,\tau_1), \tag{41}$$

where

$$Q_2(y,t)=\frac{\mu_2 y(y-t)^{-\alpha_2}}{2\Gamma(1-\alpha_2)}-\frac{\left[1+P_2(y,t)\right]a_2(y)}{\sqrt{\pi(y-t)}}, \tag{42}$$

$$Q_3(y,t) = -\frac{\mu_3(1-y)(y-t)^{-\alpha_3}}{2\Gamma(1-\alpha_3)} + \frac{[1+P_3(y,t)]\,a_3(y)}{\sqrt{\pi(y-t)}}, \tag{43}$$

$$M_j(y,t) = \frac{2(-1)^j a_j(y)}{\sqrt{\pi(y-t)}}\left[e^{-\frac{1}{4(y-t)}} + \sum_{n=1}^{+\infty} e^{-\frac{(1+2n)^2}{4(y-t)}}\right],\ (j=1,2), \tag{44}$$

$$\Phi_2(y,\tau_1) = \frac{a_2(y)}{2\sqrt{\pi y}}\int\limits_0^1 \sum_{n=-\infty}^{+\infty} e^{-\frac{4n^2+\xi^2}{4y}}\left[\frac{\xi}{y}ch4\xi n - \frac{2n}{y}sh4\xi n)\right]\tau_1(\xi)\,d\xi +$$

$$+\varphi_2'\left(\frac{y}{2}\right) + b_2(y) + \frac{1}{2\sqrt{\pi}\Gamma(1-\alpha_0)}\int\limits_0^y \frac{d\eta}{\sqrt{y-\eta}}\times$$

$$\times\int\limits_0^1 \sum_{n=-\infty}^{+\infty} e^{-\frac{4n^2+\xi^2}{4(y-\eta)}}\left[\frac{\xi}{y-\eta}ch4\xi n - \frac{2n}{y-\eta}sh4\xi n\right]\,d\xi\int\limits_0^\xi (\xi-t)^{-\alpha_0}\tau_1'(t)\,dt, \tag{45}$$

$$\Phi_3(y,\tau_1) = \frac{a_3(y)}{2\sqrt{\pi y}}\int\limits_0^1 \sum_{n=-\infty}^{+\infty} e^{-\frac{(x+2n)^2+\xi^2}{4y}}\left[\frac{\xi}{y}ch2\xi(\frac{x+2n}{2y}) - \frac{x+2n}{y}sh2\xi(\frac{x+2n}{2y})\right]\tau_1(\xi)\,d\xi +$$

$$+\varphi_3{}'\left(\frac{y+1}{2}\right) + b_3(y) + \frac{1}{2\sqrt{\pi}\Gamma(1-\alpha_0)}\int\limits_0^y \frac{d\eta}{\sqrt{y-\eta}}\times$$

$$\times\int\limits_0^1 \sum_{n=-\infty}^{+\infty} e^{-\frac{(x+2n)^2+\xi^2}{4(y-\eta)}}\left[\frac{\xi}{y-\eta}ch2\xi(\frac{x+2n}{2y}) - \frac{x+2n}{y-\eta}sh2\xi(\frac{x+2n}{2y})\right]\,d\xi\int\limits_0^\xi (\xi-t)^{-\alpha_0}\tau'_1(t)\,dt. \tag{46}$$

and using $\lim\limits_{z\to 0} z^{-\sigma}e^{-\frac{1}{z}} = 0$ for any fixed $\sigma > 0$ taking into account (11), (13), (39) we conclude that

1) the kernels $Q_j(y,t),\ \ (j=2,3)$ are continuous in $\{(y,t):\ \ 0 \le t < y \le 1\}$ and at $y \to t$ admits an estimate

$$|M_j(y,t)| \le c_0(y-t)^{-\frac{1}{2}} + c_j(y-t)^{-\alpha_j}; \tag{47}$$

2) the kernels $M_j(y,t),\ \ (j=2,3)$ are continuous and bounded in $\{(y,t):\ \ 0 \le t \le y \le 1\}$, i.e.e .

$$|M_j(y,t)| \le c_j,\quad c_0,\ c_j = const; \tag{48}$$

3) function $\Phi_j(y,\tau_1)$ belongs to the class $\Phi_j(y,\tau_1) \in C\left(\bar{J}_j\right) \cap C^2\left(J_j\right)$.

Thus, the integral equations (40) and (41) are Volterra integral equations of the second kind with a weak singularity.

According to the theory of Volterra integral equations of the second kind (Mikhlin ,1959), we conclude that the integral equation (40) is uniquely solvable in the class $C(\bar{J}_2) \cap C^2(J_2)$ and its solution is given by the formula

$$\tau_2'(y) = \Phi_2(y,\tau_1) + \int\limits_0^y M_2(y,t)\tau_3'(t)\,dt +$$

$$+\int\limits_0^y Q_2^*(y,t)\left[\Phi_2(t,\tau_1) + \int\limits_0^y M_2(t,z)\tau_3'(z)\,dz\right]dt, (0,y) \in \bar{J}_2, \tag{49}$$

where $Q_2^*(y,t)$ is the resolvent of the kernel $Q_2(y,t)$. Substituting (49) into (41), after some transformations we obtain the Volterra integral equation of the second kind with respect to the function $\tau_3^{'}(y)$:

$$\tau_3'(y) - \int_0^y \tilde{Q}_3(y,t)\tau_3'(t)\,dt = \Phi_4(y,\tau_1), (0,y) \in \bar{J}_3, \tag{50}$$

where

$$\tilde{Q}_3(y,t) = Q_3(y,t) + \int_0^y M_3(y,s)\left[M_2(s,t) + \int_t^s M_2(z,t)Q_2^*(s,z)dz\right]ds, \tag{51}$$

$$\Phi_4(y,\tau_1) = \Phi_3(y,\tau_1) + \int_0^y M_3(y,t)\left[\Phi_2(t,\tau_1) + \int_0^t \Phi_2(z,\tau_1)Q_2^*(t,z)dz\right]dt. \tag{52}$$

Using (11), (13), (39), it follows from (51) and (52) that

$$\left|\tilde{Q}_3(y,t)\right| \le c_0(y-t)^{-\frac{1}{2}} + c_3(y-t)^{-\alpha_3}, \tag{53}$$

a function $\Phi_4(y,\tau_1)$ belongs to the class $\Phi_4(y,\tau_1) \in C\left(\bar{J}_3\right) \cap C^2\left(J_3\right)$. After solving the integral equation (50), we get

$$\tau_3'(y) = \Phi_4(y,\tau_1) + \int_0^y \tilde{Q}_3^*(y,t)\Phi_4(t,\tau_1)dt, (0,y) \in \bar{J}_3, \tag{54}$$

where $\tilde{Q}_3^*(y,t)$ resolvent of the kernel $\tilde{Q}_3(y,t)$.

Using $\tau_2(0) = \tau_3(0) = 0$ from (49) and (54), respectively, we find the function $\tau_2(y)$ and $\tau_3(y)$:

$$\tau_2(y) = \int_0^y \left\{\Phi_2(t,\tau_1) + \int_0^t M_2(t,z)\Phi(z,\tau_1)dz + \right.$$

$$\left. + \int_0^t Q_2^*(t,z)\left[\Phi_2(z,\tau_1) + \int_0^z M_2(z,s)\Phi(s,\tau_1)ds\right]dz\right\}dt, (0,y) \in \bar{J}_2, \tag{55}$$

$$\tau_3(y) = \int_0^y \Phi(t,\tau_1)dt, (0,y) \in \bar{J}_3, \tag{56}$$

where

$$\Phi(t,\tau_1) = \Phi_4(t,\tau_1) + \int_0^t \tilde{Q}_3^*(t,z)\Phi_4(z,\tau_1)dz. \tag{57}$$

By supplying (49), (53) to (19) and (20) taking into account (8), (9), (11), (13), (17), (18), (55), (56) define the function $\nu_j^-(y)$ and $\nu_j^+(y)$ from the class

$$\nu_j^-(y),\ \ \nu_j^+(y) \in C\left(\bar{J}_j\right) \cap C^1\left(J_j\right), \quad (j = 2,3). \tag{58}$$

Thus, the solution of the problem A_μ can be restored in the domain Ω_0 as the solution of the first boundary value problem for the equation(1) (see (22)), and in the domains Ω_j $(j = \overline{1,3})$ as a solution of the Cauchy problem for the equation (1).

Hence, the ProblemA_μ is uniquely solvable. □

5 Conclusion

In previous works by well-known scientists, boundary value problems for loaded second-order equations of hyperbolic, parabolic, hyperbolic-parabolic and elliptic-parabolic types were studied, when the loaded part contains only the trace or derivative of the desired function. The importance of this study lies in the fact that local and non-local problems for loaded equations of mixed type of the second order with three lines of type substitution, when the loaded part contains a fractional-order integro-differential operator, have not been found. Based on this, we have investigated a local boundary value problem for a loaded parabolic-hyperbolic equation with three lines of change type, when the loaded part contains an integro-differential operator of fractional order.

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References

Abdullaev, V.M. (2020). Numerical solution of a boundary value problem for a loaded parabolic equation with nonlocal boundary conditions. *Vestnik KRAUNC. Fiz.-mat. nauki*, *32*(3), 15-28. (in Russian)

Ashurov, R., Cabada, A., & Turmetov, B. (2016). Operator method for construction of solutions of linear fractional differential equations with constant coefficients. *Fractional Calculus and Applied Analysis, 19*(1), 229-252.

Attaev, A.Kh. (2017). The Cauchy problem for the Mc Kendrick-Von Foerster loaded equation. *International Journal of Pure and Applied Mathematics*, *113*(4), 569-574.

Gekkieva, S.H. (2000). The Cauchy problem for a generalized transport equation with a time-fractional derivative. *Reports of the Adyghe (Circassian) International Academy of Sciences, 5*(1), 16-19.

Genaliev M.T. (2001). On loaded equations with periodic boundary conditions. *Differential equations*, *37*(1), 48-54.

Gorenflo, R., Luchko, Y., & Mainardi, F. (2000). Wright functions as scale-invariant solutions of the diffusion-wave equation. *Journal of Computational and Applied Mathematics, 118*(1-2), 175-191.

Islomov, B., Kuryazov, D.M. (1996). On a boundary value problem for a loaded second-order equation. *DAN RUz*, *1*(2), 3-6. (in Russian)

Islomov, B., Baltaeva, U.I. (2011). Boundary value problems for loaded differential equations of hyperbolic and mixed types of the third order. *Ufa Mathematical Journal*, *3*(3), 5-25.

Islomov, B., Baltaeva, U.I. (2015). Boundary-value problems for a third-order loaded parabolic-hyperbolic equation with variable coefficients. *Electron. J. Differ. Equ.,* 221, 1-10.

Islomov B., Kholbekov J. A.(2015a) Investigation of a boundary value problem for a loaded parabolic-hyperbolic equation with three lines of type change. Reports No. 6, 11-14.

Islomov B., Kholbekov J. A.(2015b) Boundary value problem for a loaded mixed-type equation with a fractional integro-differentiation operator. *UzMU News, 2*(1), 45-53.

Islomov B., Kholbekov J. A.(2015c) An analogue of the Tricomi problem for a loaded parabolichyperbolic equation with three lines of degeneracy – I. *Uzbek Math. J.,* 4, 47–56

Islomov B., Kholbekov J. A. (2016). An analogue of the Tricomi problem for a loaded parabolichyperbolic equation with three lines of degeneracy – II. *Uzbek Math. J.*, 1, 49–56.

Islamov, B.I., Kholbekov, J.A.(2021). On a non-local boundary value problem for a loaded parabolo-hyperbolic equation with three lines of type change. *Vestn. Sam. Gos. Tech. Un-ta. Ser. Phys.-Mat. Sciences. 25*(3), 407-422. (in Russian)

Kishin, B.S., Abdullaev, O.Kh. (2016). About a Problem for Loaded Parabolic-Hyperbolic Type Equation with Fractional Derivatives. *International Journal of Differential Equations*, *1*, Article ID 9815796, 1-6.

Jrbashyan, M.M. (1966). *Integral Transformations and Representations of Functions in the Complex Domain.* Moscow, Nauka. (in Russian)

Jrbashyan, M.M., Nersesyan, A.B. (1968). Fractional derivatives and Cauchy problems for fractional differential equations. *Izv. AN Arm. SSR. Mathematics*, *3*(1), 3-28. (in Russian)

Juraev, T.D. (1979). *Boundary value problems for equations of mixed and mixed-compound type.* FAN, 240 p.

Juraev, D.A. (2019). On a regularized solution of the Cauchy Problem for matrix factorizations of the Helmholtz equation. *Advanced Mathematical Models & Applications*, *4*(1), 86-96.

Juraev, D.A. (2020). The solution of the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation. *Advanced Mathematical Models & Applications*, *5*(2), 205-221.

Juraev, D.A., Gasimov, Y.S. (2022). On the regularization Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain. *Azerbaijon Journal of Mathematics*, *12*(1), 142-161.

Kilbas, A.A. (2004). Cauchy problem for differential equation with Caputo derivative. *Fract. Calc. Appl. Anal.*, *7*(3), 297-321.

Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and applications of fractional differential equations.* North-Holland Mathematics Studies, 204, Amsterdam: Elsevier, 523.

Kozhanov, A.I. (2004). On one non-linear loaded parabolic equation and on the related inverse problem. *Mathematical Notes, 76*(6), 840-853. (in Russian)

Khubiev, K.Yu .(2018). Boundary value problem for a loaded hyperbolic-parabolic equation with order degeneracy in the hyperbolicity domain. *Results of Science and Technology Sovr. Mat. App. Temat. Reviews*, 149, 113-117. (in Russian)

Luchko, Y., Gorenflo, R. (1999). An operational method for solving fractional differential equations with the Caputo derivatives. *Acta Math. Vietnam*, *24*(2), 207-233.

Mikhlin, S.G. (1959). *Lectures on Linear Integral Equations.* Moscow, Fizmatgiz, 232 p. (in Russian)

Nakhushev, A.M. (1983). Loaded equations and their applications. *Differential equations*, *19*(1), 86-94.(in Russian)

Nakhushev, A.M. (1995). *Equations of Mathematical Biology.* Moscow, Higher School, 301 p. (in Russian)

Nakhushev, A.M. (2006). *Problems with Displacement for Partial Differential Equations*. Moscow, Nauka, 287 p. (in Russian)

Pskhu, A.V. (2003). Solution of the first boundary value problem for the diffusion equation of fractional and continuous order. *Differential Equations*, *39*(9), 1286-1289. (in Russian)

Pskhu, A.V. (2005). Boundary value problems for partial differential equations of fractional and continuous order. *Differential Equations*, 186. (in Russian)

Shinaliev K.M., Turmetov B.H., & Umarov S.R. (2012). A fractional operator algorithm method for construction of solutions of fractional order differential equations *Fractional Calculus and Applied Analysis*, *15*(2), 267-281.

Sabitov, K.B. (2015). Initial boundary value problem for a parabolo-hyperbolic equation with loaded terms. *Izv. Vuzov. Matem.*, 6, 31-42. (in Russian)

Ubaydullayev, U.Sh. (2015). The inverse problem for a mixed loaded equation with the Riemann-Liouville operator in a rectangular domain. *Vestnik KRAUNC. Fiz.-mat. nauki*, *31*(2), 18-31. (in Russian)

Yuldashev, T.K., Islomov, B.I., & Alikulov, E.K. (2020). Initial boundary value problem for a parabolo-hyperbolic equation with loaded terms. *Lobachevskii Journal of Mathematics*, *41*(5), 922-940. (in Russian)